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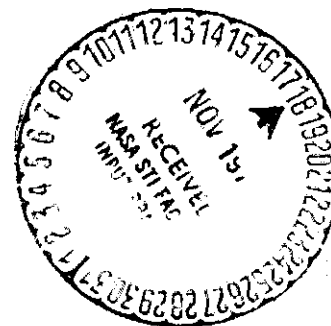
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**INVESTIGATION OF RANK 2 AND HIGHER OUTPUT
FEEDBACK FOR POLE PLACEMENT**

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INVESTIGATION OF RANK 2 AND HIGHER OUTPUT

FEEDBACK FOR POLE PLACEMENT

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ABSTRACT

One common feature of several pole placement techniques is the use of a dyadic (Rank 1) feedback matrix. The limitation of this design is examined and a design involving output feedback matrices of Rank greater than one is developed as a logical extension of the dyadic feedback design. An example is presented to illustrate the design procedure.

INTRODUCTION

Modal control has been suggested as a design tool in an effort to circumvent some of the problems in the design of multivariable systems using optimal control (Rosenbrock, 1962). In modal control the eigenvalues of the system matrix are changed to achieve the desired control objective. Much of the work which relates to the concept of modal control has been termed "pole placement." The design of linear multivariable control systems, with pole placement as a specification, has attracted the attention of several authors (Davison, 1970; Retallack and McFarlane, 1970; Fallside and Seraji, 1971; Sridhar and Lindorff, 1973). One common feature of these design methods is a dyadic feedback matrix. This paper discusses some of the disadvantages of this approach and extends the design procedure to feedback matrices of rank greater than one.

STATEMENT OF THE PROBLEM

Consider a controllable and observable linear time invariant multivariable system

$$\left. \begin{aligned} \dot{\underline{x}} &= \hat{A}\underline{x} + \hat{B}\underline{u} \\ \underline{y} &= \hat{C}\underline{x} \end{aligned} \right\} \quad (1)$$

where \underline{x} is an n vector of states, \underline{u} is an m vector of inputs and \underline{y} is a p vector of outputs. Let $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $(\rho_1, \rho_2, \dots, \rho_n)$ be the poles of the open loop and closed loop system, respectively. The problem of pole placement is to find the feedback matrix K such that the closed loop system matrix $(\hat{A} - \hat{B}K\hat{C})$ has the eigenvalues $(\rho_1, \rho_2, \dots, \rho_n)$.

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CHARACTERISTIC POLYNOMIALS

Let T be the $n \times n$ non-singular matrix of eigenvectors. We have

$$\begin{aligned} \text{open loop characteristic polynomial} &= |sI - \hat{A}| \\ &= (s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_n) \end{aligned} \quad (2)$$

and

$$\begin{aligned} \text{closed loop characteristic polynomial} &= |sI - \hat{A} + \hat{B}\hat{K}\hat{C}| \\ &= (s - \rho_1)(s - \rho_2) \dots (s - \rho_n) \end{aligned} \quad (3)$$

Further,

$$|sI - \hat{A}| = |T^{-1}| |sI - \hat{A}| |T| = |sI - T^{-1}\hat{A}T| = |sI - \Lambda| \quad (4)$$

where

$$\Lambda = \text{diag} (\lambda_1, \lambda_2, \dots, \lambda_n)$$

Also,

$$|sI - \hat{A} + \hat{B}\hat{K}\hat{C}| = |T^{-1}| |sI - \hat{A} + \hat{B}\hat{K}\hat{C}| |T| = |T^{-1}(sI - \hat{A} + \hat{B}\hat{K}\hat{C})T| = |sI - \Lambda + BKC| \quad (5)$$

where $C = \hat{C}T$ and $B = T^{-1}\hat{B}$. Define $M \triangleq BKC$. Then, it is shown in the appendix that

$$\begin{aligned} |sI - \Lambda + BKC| &= |M| + \sum_{i=1}^n (s - \lambda_i)M_i \\ &+ \sum_{i=1}^{n-1} \sum_{j=i+1}^n (s - \lambda_i)(s - \lambda_j)M_{ij} \\ &+ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n (s - \lambda_i)(s - \lambda_j)(s - \lambda_k)M_{ijk} \\ &+ \dots + \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\bar{M}_{ij} \cdot \Delta}{(s - \lambda_i)(s - \lambda_j)} + \sum_{i=1}^n \frac{m_{ii} \cdot \Delta}{s - \lambda_i} + \Delta \end{aligned} \quad (6)$$

where $M_i, M_{ij}, M_{ijk} \dots$ are the determinant of the matrices obtained by deleting the i th row and j th column, i, j rows and columns, i, j, k rows and columns, \dots , respectively. In addition, m_{ij} is the ij th element of M ,

$$\bar{M}_{ij} = \begin{vmatrix} m_{ii} & m_{ij} \\ m_{ij} & m_{jj} \end{vmatrix} \quad \text{and} \quad \Delta = (s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_n)$$

For clarity, further development is carried out in terms of a system with distinct open loop eigenvalues. The open loop and closed loop characteristic polynomials are related by the equation

$$\frac{|sI - \hat{A} + \hat{B}\hat{K}\hat{C}|}{|sI - \hat{A}|} = 1 + \sum_{i=1}^n \frac{\alpha_i}{s - \lambda_i} \quad (7)$$

The value of α_i depends on the closed loop eigenvalues (ρ_1, \dots, ρ_n). From equations (6) and (7), we have

$$\begin{aligned} \sum_{i=1}^n \frac{\alpha_i}{s - \lambda_i} &= \frac{|M|}{\Delta} + \frac{1}{\Delta} \sum_{i=1}^n (s - \lambda_i) M_i \\ &+ \frac{1}{\Delta} \sum_{i=1}^{n-1} \sum_{j=i+1}^n (s - \lambda_i)(s - \lambda_j) M_{ij} \\ &+ \frac{1}{\Delta} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n (s - \lambda_i)(s - \lambda_j)(s - \lambda_k) M_{ijk} \\ &+ \dots + \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\bar{M}_{ij}}{(s - \lambda_i)(s - \lambda_j)} + \sum_{i=1}^n \frac{m_{ii}}{s - \lambda_i} \end{aligned} \quad (8)$$

Note that $|M| = 0$ except for $m = p = n$.

DIADIC K MATRIX

Results for this case can be obtained easily as a special case of equation (8). When K has rank one, that is,

$$K = \underline{f} \underline{d}^T \quad (9)$$

$$m_{ij} = b_i K c_j = \underline{b}_i \underline{f} \underline{d}^T \underline{c}_j = \underline{d}^T \underline{c}_j \underline{b}_i \underline{f} \quad (10)$$

where $\underline{f} = [f_1, f_2, \dots, f_m]^T$, $\underline{d}^T = [d_1, d_2, \dots, d_p]$, \underline{b}_i is the i th row of B and \underline{c}_i is the i th column of C . With this choice of K

$$|M|, M_{ii}, M_{ij}, M_{ijk}, \dots, \bar{M}_{ij} = 0 \quad (11)$$

Now equation (8) reduces to

$$\sum_{i=1}^n \frac{\alpha_i}{s - \lambda_i} = \sum_{i=1}^n \frac{m_{ii}}{s - \lambda_i} \quad (12)$$

Then,

$$\alpha_i = m_{ii} = \underline{d}^T \underline{c}_i \underline{b}_i \underline{f} \quad (13)$$

Equation (13) is the same as equation (9) in Sridhar and Lindorff (1973) and K can be found following the procedure outlined by them.

It is important to recall that the simplicity of the above pole-shifting algorithm is a consequence of K being a dyadic. However, this results in a loss of design freedom available in multivariable design problems. In other words, $K = \underline{f} \underline{d}^T$ maps any output vector \underline{y} into a vector proportional to \underline{f} . As \underline{y} varies the range of possible controls varies along the line \underline{f} instead of the range of \underline{y} . This is illustrated for $m = p = 2$ in figure 1. The effect of this is that the feedback will couple all the modes of the system although some of the modes of the system may be open loop decoupled and it may be difficult to satisfy performance criteria other than a simple allocation of the closed loop poles. In an effort to overcome some of these difficulties the design is extended to feedback matrices of higher rank in the next section.

FEEDBACK MATRIX OF RANK 2

In this case the determinants of all $\ell \times \ell$ matrices formed from M with $\ell > 2$ are equal to zero and equation (8) reduces to

$$\sum_{i=1}^n \frac{\alpha_i}{s - \lambda_i} = \frac{|M|}{\Delta} + \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\bar{M}_{ij}}{(s - \lambda_i)(s - \lambda_j)} + \sum_{i=1}^n \frac{m_{ii}}{s - \lambda_i} \quad (14)$$

Next,

$$\frac{|M|}{\Delta} = \sum_{i=1}^n \frac{1}{\prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_i - \lambda_j)} \cdot \frac{|M|}{s - \lambda_i} \quad (15)$$

We have

$$\begin{aligned} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\bar{M}_{ij}}{(s-\lambda_i)(s-\lambda_j)} &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n \bar{M}_{ij} \left[\frac{1/(\lambda_i - \lambda_j)}{s-\lambda_i} + \frac{1/(\lambda_j - \lambda_i)}{s-\lambda_j} \right] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\bar{M}_{ij}}{\lambda_i - \lambda_j} \left(\frac{1}{s-\lambda_i} - \frac{1}{s-\lambda_j} \right) \end{aligned} \quad (16)$$

By expanding the summation on the R.H.S. of (16) and regrouping the terms,

$$\begin{aligned} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\bar{M}_{ij}}{(s-\lambda_i)(s-\lambda_j)} &= \frac{1}{s-\lambda_1} \left[\frac{\bar{M}_{12}}{\lambda_1-\lambda_2} + \frac{\bar{M}_{13}}{\lambda_1-\lambda_3} + \dots + \frac{\bar{M}_{1,n-1}}{\lambda_1-\lambda_{n-1}} + \frac{\bar{M}_{1n}}{\lambda_1-\lambda_n} \right] \\ &\quad + \frac{1}{s-\lambda_2} \left[-\frac{\bar{M}_{12}}{\lambda_1-\lambda_2} + \frac{\bar{M}_{23}}{\lambda_2-\lambda_3} + \dots + \frac{\bar{M}_{2,n-1}}{\lambda_2-\lambda_{n-1}} + \frac{\bar{M}_{2n}}{\lambda_2-\lambda_n} \right] \\ &\quad + \frac{1}{s-\lambda_3} \left[-\frac{\bar{M}_{13}}{\lambda_1-\lambda_3} - \frac{\bar{M}_{23}}{\lambda_2-\lambda_3} + \frac{\bar{M}_{34}}{\lambda_3-\lambda_4} + \dots + \frac{\bar{M}_{3n}}{\lambda_3-\lambda_n} \right] \\ &\quad \vdots \\ &\quad + \frac{1}{s-\lambda_{n-1}} \left[-\frac{\bar{M}_{1,n-1}}{\lambda_1-\lambda_{n-1}} - \frac{\bar{M}_{2,n-1}}{\lambda_2-\lambda_{n-1}} - \dots - \frac{\bar{M}_{n-2,n-1}}{\lambda_{n-2}-\lambda_{n-1}} + \frac{\bar{M}_{n-1,n}}{\lambda_{n-1}-\lambda_n} \right] \\ &\quad + \frac{1}{s-\lambda_n} \left[-\frac{\bar{M}_{1n}}{\lambda_1-\lambda_n} - \frac{\bar{M}_{2n}}{\lambda_2-\lambda_n} - \dots - \frac{\bar{M}_{n-2,n-1}}{\lambda_{n-2}-\lambda_n} - \frac{\bar{M}_{n-1,n}}{\lambda_{n-1}-\lambda_n} \right] \end{aligned} \quad (17)$$

observing the fact that

$$\bar{M}_{ij} = \bar{M}_{ji} = \begin{vmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{vmatrix}$$

equation (17) becomes

$$\begin{aligned}
\sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\bar{M}_{ij}}{(s - \lambda_i)(s - \lambda_j)} &= \frac{1}{s - \lambda_1} \sum_{i=2}^n \frac{\bar{M}_{1i}}{\lambda_1 - \lambda_i} + \frac{1}{s - \lambda_2} \sum_{\substack{i=1 \\ i \neq 2}}^n \frac{\bar{M}_{2i}}{\lambda_2 - \lambda_i} \\
&\quad + \dots + \frac{1}{s - \lambda_n} \sum_{i=1}^{n-1} \frac{\bar{M}_{ni}}{\lambda_n - \lambda_i} \\
&= \sum_{i=1}^n \frac{1}{s - \lambda_i} \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\bar{M}_{ij}}{\lambda_i - \lambda_j}
\end{aligned} \tag{18}$$

Combining equations (14), (15), and (18) we have

$$\sum_{i=1}^n \frac{\alpha_i}{s - \lambda_i} = \sum_{i=1}^n \frac{1}{\prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_i - \lambda_j)} \cdot \frac{|M|}{s - \lambda_i} + \sum_{i=1}^n \frac{1}{s - \lambda_i} \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\bar{M}_{ij}}{\lambda_i - \lambda_j} + \sum_{i=1}^n \frac{m_{ii}}{s - \lambda_i}$$

Equating coefficients of $(s - \lambda_i)^{-1}$,

$$\alpha_i = \frac{|M|}{\prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_i - \lambda_j)} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\bar{M}_{ij}}{\lambda_i - \lambda_j} + m_{ii}, \quad i = 1, 2, \dots, n \tag{19}$$

For $n > 2$, equation (19) reduces to

$$\alpha_i = \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\bar{M}_{ij}}{\lambda_i - \lambda_j} + m_{ii}, \quad i = 1, 2, \dots, n \tag{20}$$

K can be decomposed as the product of two matrices G and H, that is,
 $K = GH$ where G is a $(m \times 2)$ matrix and H is a $(2 \times p)$ matrix. Further,

$$G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix}, \quad H = [\underline{h}_1 \quad \underline{h}_2 \quad \dots \quad \underline{h}_p], \quad \underline{g}_i = [g_{i1} \quad g_{i2}], \quad \underline{h}_i = \begin{bmatrix} h_{1i} \\ h_{2i} \end{bmatrix}$$

$$K = \begin{bmatrix} g_{1h1} & g_{1h2} & \dots & g_{1hp} \\ \vdots & & & \\ g_{mh1} & g_{mh2} & \dots & g_{mhp} \end{bmatrix}$$

and $m_{ij} = b_i K_{cj} = b_i G H_{cj}$. Now, we have to solve the n nonlinear equations (20) in $2(m+p)$ variables for pole placement using output feedback matrices of Rank 2.

Example

This simple example illustrates the differences between the dyadic and Rank 2 method of pole placement. Consider the system

$$\dot{\underline{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underline{x}$$

The system transfer function is

$$\underline{y} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{s+3} & \frac{2s+4}{(s+1)(s+3)} \end{bmatrix} \underline{u}$$

The system is observable and controllable with $m = 2$, $p = 2$, and $n = 3$. The output y_1 depends only on the input u_1 . The open loop poles are at 1, -1, and -3. It is desired to stabilize the system while maintaining that y_1 depends only on u_1 .

Max $(m,p) = 2$ poles can be placed using the output feedback. Let the closed loop poles be at $-\rho_1 = -0.5$, $-\rho_2 = -1.5$ while the location of the third pole $-\rho_3$ depends on $(-\rho_1, -\rho_2)$. For this choice of closed loop poles,

$$\alpha_1 = \frac{15}{32} (1 + \rho_3)$$

$$\alpha_2 = \frac{2}{32} (2 + \rho_3)$$

$$\alpha_3 = \frac{15}{32} (-3 + \rho_3)$$

For a dyadic feedback

$$K = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} [d_1 d_2] = \begin{bmatrix} f_1 d_1 & f_1 d_2 \\ f_2 d_1 & f_2 d_2 \end{bmatrix}$$

Since

$$\underline{d}^T \underline{c}_i \underline{h}_i \underline{f} = \alpha_i, \quad i = 1, 2, 3$$

We have

$$d_1 f_1 = -3/2(1 + \rho_3)$$

$$d_2 f_2 = 12/5(2 + \rho_3)$$

$$d_2(f_1 + f_2) = 1/10(\rho_3 - 3)$$

Solving these equations with $d_1 = 1$ and $d_2 = -2$, we get

$$f_1 = 0.6104 \quad \text{and} \quad f_2 = 0.0218$$

For this choice of

$$K = \begin{bmatrix} 0.6104 & -1.2208 \\ 0.0218 & -0.0436 \end{bmatrix}$$

two poles can be placed at $(-0.5, -1.5)$ and the third pole is located at -0.3023 . However, it is not possible to satisfy the second criterion that y_1 depend only on u_1 as can be seen from the closed loop system matrix

$$A - BKC = \begin{bmatrix} 1 - f_1 d_1 & -f_1 d_2 & -f_1 d_2 \\ -f_2 d_1 & -1 - f_2 d_2 & -f_2 d_2 \\ -(f_1 + f_2) d_1 & -(f_1 + f_2) d_2 & -3 - (f_1 + f_2) d_2 \end{bmatrix}$$

Next consider K with Rank 2 which can be written as

$$K = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} \underline{g}_1 \\ \underline{g}_2 \end{bmatrix} [\underline{h}_1 \quad \underline{h}_2]$$

where $\underline{g}_1 = [f_{11} \quad f_{12}]$, $\underline{g}_2 = [f_{21} \quad f_{22}]$, $\underline{h}_1 = [d_{11} \quad d_{21}]^T$ and $\underline{h}_2 = [d_{12} \quad d_{22}]^T$. The closed loop system matrix with the new feedback is given by

$$A - BKC = \begin{bmatrix} 1 - \underline{g}_1 \underline{h}_1 & -\underline{g}_1 \underline{h}_2 & -\underline{g}_1 \underline{h}_2 \\ -\underline{g}_2 \underline{h}_1 & -1 - \underline{g}_2 \underline{h}_2 & -\underline{g}_2 \underline{h}_2 \\ -(\underline{g}_1 + \underline{g}_2) \underline{h}_1 & -(\underline{g}_1 + \underline{g}_2) \underline{h}_2 & -3 - (\underline{g}_1 + \underline{g}_2) \underline{h}_2 \end{bmatrix}$$

If y_1 should depend only on u_1 then

$$-\underline{g}_1 \underline{h}_2 = 0$$

From (20), for pole placement we have to satisfy the equations

$$\sum_{\substack{j=1 \\ j \neq i}}^3 \frac{\bar{M}_{ij}}{\lambda_i - \lambda_j} + m_{ii} = \alpha_i, \quad i = 1, 2, 3$$

These equations with the condition $-\underline{g}_1 \underline{h}_2 = 0$ reduce to

$$\begin{aligned} \frac{3}{4} \underline{g}_1 \underline{h}_1 + \underline{g}_2 \underline{h}_2 + \underline{g}_1 \underline{h}_1 &= \alpha_1 \\ -\frac{1}{2} \underline{g}_1 \underline{h}_1 + \underline{g}_2 \underline{h}_2 + \underline{g}_2 \underline{h}_2 &= \alpha_2 \\ -\frac{1}{4} \underline{g}_1 \underline{h}_1 + \underline{g}_2 \underline{h}_2 + \underline{g}_2 \underline{h}_2 &= \alpha_3 \end{aligned}$$

As in the dyadic case, the choice of closed loop poles at $(-0.5, -1.5)$ require that $\alpha_1 = (15/32)(1 + \rho_3)$, $\alpha_2 = (2/32)(2 + \rho_3)$ and $\alpha_3 = (15/32)(-3 + \rho_3)$. Noticing the fact that $\underline{g}_1 \underline{h}_1 = k_{11}$, $\underline{g}_2 \underline{h}_2 = k_{22}$ and $\underline{g}_1 \underline{h}_2 = k_{12}$, the feedback matrix has to be chosen subject to

$$k_{12} = 0$$

$$k_{11} + \frac{3}{4} k_{11} \cdot k_{22} = \frac{15}{32} (\rho_3 + 1)$$

$$k_{22} - \frac{1}{2} k_{11} \cdot k_{22} = \frac{2}{32} (\rho_3 - 1)$$

$$k_{22} - \frac{1}{4} k_{11} \cdot k_{22} = \frac{15}{32} (\rho_3 - 3)$$

Solving these equations

$$K = \begin{bmatrix} \frac{3}{2} & 0 \\ k_{21} & \frac{3}{4} \end{bmatrix}$$

and the third pole is located at -4. With $k_{21} = -3/2$, the closed loop system transfer function is

$$Y = \begin{bmatrix} \frac{1}{s + 0.5} & 0 \\ \frac{s^2 + 1.55 - 5.5}{(s + 0.5)(s + 1.5)(s + 4)} & \frac{2s + 7}{(s + 1.5)(s + 4)} \end{bmatrix} \underline{u}$$

and the closed loop system meets the design requirements.

OTHER METHODS

Feedback matrices of higher rank can be easily computed for the pole placement of multivariable systems using state feedback by Kalman's (1971) method. The given system is transformed to the Luenberger canonical form (Jordan and Sridhar, 1973). Let $\sigma_1, \sigma_2, \dots, \sigma_m$ be the controllability indices of the system. Then the transformed system is given by

$$\dot{\underline{\xi}} = A\underline{\xi} + B\underline{u}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ A_{m,1} & A_{m2} & \dots & A_{m,m} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} \\ B_{12} \\ \vdots \\ B_{1m} \end{bmatrix}$$

The dimensions of A_{ij} and B_{1i} and $(\sigma_i \times \sigma_j)$ and $(\sigma_m \times m)$, respectively. Further,

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ * & * & * & \dots & * \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \dots & * \end{bmatrix}$$

and

$$B_{1i} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & 1 & * & * \end{bmatrix}$$

+ ith column

The * denotes a nonzero element. From a study of the Luenberger's canonical form it is evident that the feedback affects only the $\sigma_1, (\sigma_1 + \sigma_2), \dots, (\sigma_1 + \sigma_2 + \dots, \sigma_m)$ rows of the system matrix. Let A_m , B_m , and A_d consist of the $\sigma_1, (\sigma_1 + \sigma_2), \dots, (\sigma_1 + \sigma_2 + \dots, \sigma_m)$ rows of A , B , and $(A + BK)$, respectively. Then,

$$A_d = A_m + B_m K \quad (21)$$

and

$$K = B_m^{-1} (A_d - A_m) \quad (22)$$

B_m^{-1} exists since $\det B_m = 1$. In particular, if we choose

$$A_d = \begin{bmatrix} 0 & \dots & 1 & \dots & \dots & 0 \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \\ 0 & \dots & 0 & \dots & \dots & 1 & 0 \\ -a_0 - a_1 & \dots & \dots & \dots & \dots & -a_{n-1} \end{bmatrix}$$

where the last row corresponds to the coefficient of the closed loop characteristic polynomial

$$(s - p_1)(s - p_2) \dots (s - p_n) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0$$

and the 1's appear in the $\sigma_1, (\sigma_1 + \sigma_2), \dots, (\sigma_1 + \sigma_2 + \dots, \sigma_{m-1})$ columns. With this, A_d ,

$$A + BK = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 1 \\ -a_0 - a_1 & \dots & \dots & -a_{n-1} \end{bmatrix}$$

has eigenvalues at (p_1, p_2, \dots, p_n) and K is given by (22).

The above constructive procedure reduces to

$$KC = B_m^{-1}(A_d - A_m) \quad (23)$$

for pole placement by output feedback. Munro (1973) has given conditions under which equation (23) can be solved to obtain arbitrary assignment of all the poles using constant output feedback.

Sankaran (1974) has suggested an iterative procedure for determining the constant gain output feedback matrix that will stabilize a system. If the system matrix A varies to $A + \delta A$, then the corresponding variation in the eigenvalue is given by (Rosenbrock, 1965)

$$\delta\lambda_r = \frac{\text{trace}[Q \cdot \delta A]}{\text{trace}[Q]} \quad (24)$$

where

$$Q = \prod_{\substack{i=1 \\ i \neq r}}^n (A - \lambda_i I)$$

Assuming that δA results due to output feedback and denoting the gain matrix by δK , then

$$\delta A = B \cdot \delta K \cdot C$$

and a linear equation can be written for the elements of the gain matrix. Since the equation (24) is valid only for small perturbations, variations in the eigenvalues should be applied in small steps in the desired direction to compute δK .

The relative merits of the Jordan canonical form approach to pole placement outlined in this paper and the Kalman approach to pole placement are discussed in the next few paragraphs. Godbout (1974) has made an extensive comparison of the two methods.

The Kalman method of pole placement uses the Luenberger Canonical Form of a system to shift the modes of that system. However, a system must be completely controllable in order to be transformed to Luenberger form. This restriction implies that the Kalman algorithm cannot be applied to any system that has at least one uncontrollable mode. This is not the case with the Jordan Canonical Form approach to pole placement. This algorithm allows the shifting of any or all of the controllable modes of a system that may also contain uncontrollable modes. This is a significant advantage for the Jordan Canonical Form approach over the Kalman approach.

The control over pole motion is probably the most desirable feature of the Jordan Canonical Form scheme. The method allows the designer to choose the closed-loop value to which a specified open-loop mode will be shifted. The Kalman algorithm does not have this feature. The pole motion cannot be chosen or even determined when using this method.

The Kalman pole placement scheme does have one desirable characteristic in that it is very straightforward. Furthermore, the Kalman method does not require knowledge of the eigenvalues of the system, whereas the Jordan Canonical Form approach needs an accurate estimate of the eigenvalues in order to construct the Jordan form of the system properly. These eigenvalue estimates are generated by Francis' QR method which sometimes has difficulty converging when a system has repeated roots.

CONCLUSIONS

This paper considers the problem of pole placement in multivariable systems using output feedback matrices of Rank greater than one with particular reference to matrices of Rank 2. The results have been developed for a system with distinct eigenvalues. However, they can be easily extended to systems with multiple eigenvalues. The feedback matrices of higher rank provide considerable design freedom. But, the resulting equations for the feedback matrix are more complex than in the dyadic case.

APPENDIX

Theorem: Let Ω be a diagonal matrix with elements $(\Omega_1, \Omega_2, \dots, \Omega_n)$ and let M be a $n \times n$ matrix. Then

$$\begin{aligned} |\Omega + M| &= |M| + \sum_{i=1}^n \Omega_i M_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^n \Omega_i \Omega_j M_{ij} \\ &\quad + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n \Omega_i \Omega_j \Omega_k M_{ijk} + \dots \\ &\quad + \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\bar{M}_{ij} \cdot \Delta}{\Omega_i \cdot \Omega_j} + \sum_{i=1}^n \frac{m_{ii} \cdot \Delta}{\Omega_i} + \Delta \end{aligned}$$

where M_i , M_{ij} , M_{ijk} . . . are the determinant of the matrices obtained by deleting the i th row and column, i, j rows and columns, i, j, k rows and columns, . . . , respectively. In addition,

$$\bar{M}_{ij} = \begin{vmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{vmatrix}$$

and $\Delta = \Omega_1 \Omega_2 \dots \Omega_n$.

Proof: The proof depends on the repeated application of the following Lemma.

Lemma (Stiab, 1969)

Let P , Q , and R be three $n \times n$ matrices identical except for their p th rows. Further, Let the p th row of R be the vector sum of the p th rows of P and Q . Then,

$$|R| = |P| + |Q| \tag{A1}$$

Define

$$M^{\ell} \triangleq \begin{bmatrix} \Omega_1 + m_{11} & m_{12} & \dots & m_{1\ell} & \dots & m_{1n} \\ m_{12} & \Omega_2 + m_{22} & \dots & m_{2\ell} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ m_{\ell 1} & m_{\ell 2} & \dots & \Omega_{\ell} + m_{\ell\ell} & \dots & m_{\ell n} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & & m_{n\ell} & & m_{nn} \end{bmatrix}$$

and $M_{12} \dots n \triangleq 1$. From the lemma,

$$M^1 = \begin{bmatrix} \Omega_1 + m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} = |M| + \begin{bmatrix} \Omega_1 & 0 & \dots & 0 \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix}$$

that is,

$$M^1 = |M| + \Omega_1 M_1 \quad (A2)$$

Again from equations (A1) and (A2)

$$M^2 = \begin{bmatrix} \Omega_1 + m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & \Omega_2 + m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} = M^1 + \begin{bmatrix} \Omega_1 + m_{11} & m_{12} & \dots & m_{1n} \\ 0 & \Omega_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix}$$

that is,

$$\begin{aligned} M^2 &= M^1 + \Omega_2 [M_2 + \Omega_1 M_{12}] \\ &= |M| + \sum_{i=1}^2 \Omega_i M_i + \Omega_1 \Omega_2 M_{12} \end{aligned} \quad (A3)$$

Next,

$$\begin{aligned}
 M^3 &= M^2 + \begin{bmatrix} \Omega_1 + m_{11} & m_{12} & m_{13} & \dots & m_{1n} \\ m_{21} & \Omega_2 + m_{22} & m_{23} & \dots & m_{2n} \\ 0 & 0 & \Omega_3 & \dots & 0 \\ \vdots & & & & \vdots \\ m_{n1} & m_{n2} & m_{n3} & \dots & m_{nn} \end{bmatrix} \\
 &= |M| + \sum_{i=1}^3 \Omega_i M_i + \sum_{i=1}^2 \sum_{j=i+1}^3 \Omega_i \Omega_j M_{ij} + \Omega_1 \Omega_2 \Omega_3 M_{123} \quad (A4)
 \end{aligned}$$

Applying the lemma successively to M^4, M^5, \dots, M^n

$$\begin{aligned}
 M^n &= |M| + \sum_{i=1}^n \Omega_i M_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^n \Omega_i \Omega_j M_{ij} + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n \Omega_i \Omega_j \Omega_k M_{ijk} \\
 &\quad + \dots + \sum_{i=1}^3 \sum_{j=i+1}^4 \dots \sum_{p=1}^{n-1} \sum_{q=p+1}^n \Omega_i \Omega_j \dots \Omega_p \Omega_q \Omega_r M_{ijk \dots pq} \\
 &\quad + \sum_{i=1}^2 \sum_{j=i+1}^3 \dots \sum_{q=1}^{n-1} \sum_{r=q+1}^n \Omega_i \Omega_j \dots \Omega_p \Omega_q \Omega_r M_{ijk \dots pqr} + \Delta
 \end{aligned}$$

$M_{ijk \dots pq}$ is the matrix with $(n-1)$ rows and columns deleted, that is, it is one of the diagonal elements of M . Also,

$$\Omega_1 \Omega_2 \dots \Omega_{i-1} \Omega_{i+1} \dots \Omega_n = \frac{\Delta}{\Omega_i}$$

Hence,

$$\sum_{i=1}^2 \sum_{j=i+1}^3 \dots \sum_{q=1}^{n-1} \sum_{r=q+1}^n \Omega_i \Omega_j \Omega_k \dots \Omega_p \Omega_q \Omega_r M_{ijk \dots pqr} = \sum_{i=1}^n \frac{m_{ii} \cdot \Delta}{\Omega_i}$$

Recalling the definition of \bar{M}_{ij} ,

$$\sum_{i=1}^3 \sum_{j=i+1}^4 \dots \sum_{p=1}^{n-1} \sum_{q=p+1}^n \Omega_i \Omega_j \dots \Omega_p \Omega_q M_{ijk} \dots p q = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\bar{M}_{ij}}{\Omega_i \Omega_j} \cdot \Delta$$

Now we can rewrite

$$\begin{aligned} |\Omega + M| &= M^n = |M| + \sum_{i=1}^n \Omega_i M_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^n \Omega_i \Omega_j M_{ij} \\ &+ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n \Omega_i \Omega_j \Omega_k M_{ijk} + \dots \\ &+ \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\bar{M}_{ij} \cdot \Delta}{\Omega_i \Omega_j} + \sum_{i=1}^n \frac{m_{ii} \cdot \Delta}{\Omega_i} + \Delta \end{aligned}$$

This proves the theorem.

Closed Loop Characteristic Polynomial:

$$|sI - A + BKC|$$

This is a special case of the theorem with

$$\Omega_i = (s - \lambda_i), \quad i = 1, 2, \dots, n \quad \text{and} \quad M = BKC$$

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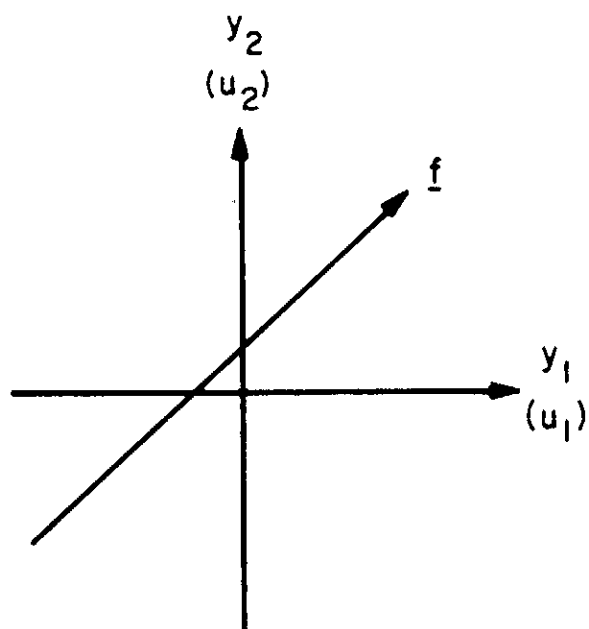


Figure 1.- Range of dyadic control ($m = p = 2$).